Exercise 2.5.2

Consider $u(x, y)$ satisfying Laplace's equation inside a rectangle ($0 < x < L$, $0 < y < H$) subject to the boundary conditions

$$\frac{\partial u}{\partial x}(0, y) = 0, \qquad \frac{\partial u}{\partial y}(x, 0) = 0$$
$$\frac{\partial u}{\partial x}(L, y) = 0, \qquad \frac{\partial u}{\partial y}(x, H) = f(x).$$

(a) *Without* solving this problem, briefly explain the physical condition under which there is a solution to this problem.

(b) Solve this problem by the method of separation of variables. Show that the method works only under the condition of part (a). [*Hint*: You may use (2.5.16) without derivation.]

(c) The solution [part (b)] has an arbitrary constant. Determine it by consideration of the time-dependent heat equation (1.5.11) subject to the initial condition

$$u(x, y, 0) = g(x, y).$$

Solution

Part (a)

The Laplace equation is

$$\nabla^2 u = 0.$$

Integrate both sides over the area A of the rectangle in the xy-plane.

$$\iint_A \nabla^2 u \, dA = 0$$

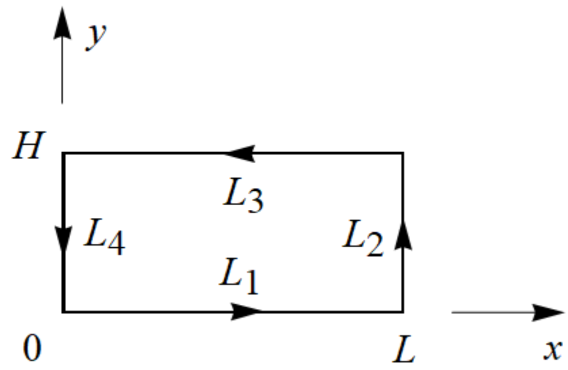
$$\iint_A \nabla \cdot \nabla u \, dA = 0$$

Apply Green's theorem (essentially the divergence theorem in two dimensions) to turn this area integral into a counterclockwise line integral over the area's boundary.

$$\oint_{\text{bdy } A} \nabla u \cdot \mathbf{n} \, dr = 0$$

Here $\mathbf{n}$ represents a unit vector normal to the integration path. Since the domain is a rectangle, the closed loop integral is the sum of four integrals—one over each line segment.

$$\int_{L_1} \nabla u \cdot \mathbf{n} \, dr + \int_{L_2} \nabla u \cdot \mathbf{n} \, dr + \int_{L_3} \nabla u \cdot \mathbf{n} \, dr + \int_{L_4} \nabla u \cdot \mathbf{n} \, dr = 0$$



$$\int_0^L \frac{\partial u}{\partial y}(x,0)\,dx + \int_0^H \frac{\partial u}{\partial x}(L,y)\,dy + \int_L^0 \frac{\partial u}{\partial y}(x,H)\,dx + \int_H^0 \frac{\partial u}{\partial x}(0,y)\,dy = 0$$

Substitute each of the boundary conditions.

$$\int_0^L (0)\,dx + \int_0^H (0)\,dy + \int_L^0 f(x)\,dx + \int_H^0 (0)\,dy = 0$$

$$-\int_0^L f(x)\,dx = 0$$

Therefore, the solvability condition for this problem is

$$\int_0^L f(x)\,dx = 0.$$

This implies that for there to be a solution, no net heat can enter at the $y = H$ edge of the rectangle. Each of the other edges is insulated, so if the solvability condition is not satisfied, the temperature u in the area will diverge.

Part (b)

Because Laplace's equation and all but one of the boundary conditions are linear and homogeneous, the method of separation of variables can be applied. Assume a product solution of the form $u(x,y) = X(x)Y(y)$ and substitute it into the PDE

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

and the homogeneous boundary conditions.

$$\frac{\partial u}{\partial x}(0,y) = 0 \quad \rightarrow \quad X'(0)Y(y) = 0 \quad \rightarrow \quad X'(0) = 0$$

$$\frac{\partial u}{\partial x}(L,y) = 0 \quad \rightarrow \quad X'(L)Y(y) = 0 \quad \rightarrow \quad X'(L) = 0$$

$$\frac{\partial u}{\partial y}(x,0) = 0 \quad \rightarrow \quad X(x)Y'(0) = 0 \quad \rightarrow \quad Y'(0) = 0$$

Separate variables in the PDE.

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0$$

Bring the second term to the right side. (Note that the final answer will be the same regardless of which side the minus sign is on.)

$$\underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{\text{function of } x} = \underbrace{-\frac{1}{Y}\frac{d^2Y}{dy^2}}_{\text{function of } y}$$

The only way a function of x can be equal to a function of y is if both are equal to a constant λ.

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} = \lambda$$

As a result of applying the method of separation of variables, the PDE has reduced to two ODEs—one in x and one in y.

$$\left.\begin{aligned} \frac{1}{X}\frac{d^2X}{dx^2} &= \lambda \\ -\frac{1}{Y}\frac{d^2Y}{dy^2} &= \lambda \end{aligned}\right\}$$

Values of λ for which nontrivial solutions of these equations exist are called the eigenvalues, and the solutions themselves are known as the eigenfunctions. We will solve the ODE for X first since there are two boundary conditions for it. Suppose first that λ is positive: $\lambda = \alpha^2$. The ODE for X becomes

$$X'' = \alpha^2 X.$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$X(x) = C_1 \cosh \alpha x + C_2 \sinh \alpha x$$

Take a derivative of it.

$$X'(x) = \alpha(C_1 \sinh \alpha x + C_2 \cosh \alpha x)$$

Apply the boundary conditions to determine C_1 and C_2.

$$\begin{aligned} X'(0) &= \alpha(C_2) = 0 \\ X'(L) &= \alpha(C_1 \sinh \alpha L + C_2 \cosh \alpha L) = 0 \end{aligned}$$

The first equation implies that C_2, so the second one reduces to $C_1 \alpha \sinh \alpha L = 0$. No nonzero value of α satisfies this equation, so C_1 must be zero. The trivial solution is obtained, so there are no positive eigenvalues. Suppose secondly that λ is zero: $\lambda = 0$. The ODE for X becomes

$$X'' = 0.$$

Integrate both sides with respect to x.

$$X' = C_3$$

Apply the boundary conditions to determine C_3.

$$\begin{aligned} X'(0) &= C_3 = 0 \\ X'(L) &= C_3 = 0 \end{aligned}$$

Consequently,

$$X' = 0.$$

Integrate both sides with respect to x once more.

$$X(x) = C_4$$

Because $X(x)$ is nonzero, zero is an eigenvalue; the eigenfunction associated with it is $X_0(x) = 1$. With this value for λ, solve the ODE for Y.

$$Y'' = 0$$

Integrate both sides with respect to y.

$$Y' = C_5$$

Apply the boundary condition to determine one of the constants.

$$Y'(0) = C_5 = 0$$

So then

$$Y' = 0.$$

Integrate both sides with respect to y once more.

$$Y(y) = C_6$$

Suppose thirdly that λ is negative: $\lambda = -\beta^2$. The ODE for X becomes

$$X'' = -\beta^2 X.$$

The general solution is written in terms of sine and cosine.

$$X(x) = C_7 \cos \beta x + C_8 \sin \beta x$$

Take a derivative of it.

$$X'(x) = \beta(-C_7 \sin \beta x + C_8 \cos \beta x)$$

Apply the boundary conditions to determine C_7 and C_8.

$$\begin{aligned} X'(0) &= \beta(C_8) = 0 \\ X'(L) &= \beta(-C_7 \sin \beta L + C_8 \cos \beta L) = 0 \end{aligned}$$

The first equation implies that $C_8 = 0$, so the second one reduces to $-C_7 \beta \sin \beta L = 0$. To avoid getting the trivial solution, we insist that $C_7 \neq 0$. Then

$$\begin{aligned} -\beta \sin \beta L &= 0 \\ \sin \beta L &= 0 \\ \beta L &= n\pi, \quad n = 1, 2, \ldots \\ \beta_n &= \frac{n\pi}{L}. \end{aligned}$$

There are negative eigenvalues $\lambda = -n^2\pi^2/L^2$, and the eigenfunctions associated with them are

$$\begin{aligned} X(x) &= C_7 \cos \beta x + C_8 \sin \beta x \\ &= C_7 \cos \beta x \quad \rightarrow \quad X_n(x) = \cos \frac{n\pi x}{L}. \end{aligned}$$

With this formula for λ, solve the ODE for Y now.

$$\frac{d^2Y}{dy^2} = \frac{n^2\pi^2}{L^2}Y$$

The general solution is written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_9 \cosh\frac{n\pi y}{L} + C_{10}\sinh\frac{n\pi y}{L}$$

Take a derivative of it.

$$Y'(y) = \frac{n\pi}{L}\left(C_9 \sinh\frac{n\pi y}{L} + C_{10}\cosh\frac{n\pi y}{L}\right)$$

Use the boundary condition to determine one of the constants.

$$Y'(0) = \frac{n\pi}{L}(C_{10}) = 0 \quad \rightarrow \quad C_{10} = 0$$

So then

$$Y(y) = C_9 \cosh\frac{n\pi y}{L} \quad \rightarrow \quad Y_n(y) = \cosh\frac{n\pi y}{L}.$$

According to the principle of superposition, the general solution to the PDE for u is a linear combination of $X(x)Y(y)$ over all the eigenvalues.

$$\boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cosh\frac{n\pi y}{L}}$$

Use the final inhomogeneous boundary condition $\frac{\partial u}{\partial y}(x,H) = f(x)$ to determine A_n. Take a derivative of the solution with respect to y.

$$\frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L}\cos\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}$$

Apply the boundary condition.

$$\frac{\partial u}{\partial y}(x,H) = \sum_{n=1}^{\infty} A_n \frac{n\pi}{L}\sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L} = f(x)$$

Multiply both sides by $\cos(m\pi x/L)$, where m is an integer,

$$\sum_{n=1}^{\infty} A_n \frac{n\pi}{L}\sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L} = f(x)\cos\frac{m\pi x}{L}$$

and then integrate both sides with respect to x from 0 to L.

$$\int_0^L \sum_{n=1}^{\infty} A_n \frac{n\pi}{L}\sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Bring the constants in front of the integral on the left.

$$\sum_{n=1}^{\infty} A_n \frac{n\pi}{L}\sinh\frac{n\pi H}{L}\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \int_0^L f(x)\cos\frac{m\pi x}{L}\,dx$$

Because the cosine functions are orthogonal, the integral on the left is zero if $n \neq m$. As a result, every term in the infinite series vanishes except for the $n = m$ one.

$$A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \int_0^L \cos^2 \frac{n\pi x}{L}\, dx = \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

$$A_n \frac{n\pi}{L} \sinh \frac{n\pi H}{L} \left(\frac{L}{2}\right) = \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx$$

Therefore,

$$\boxed{A_n = \frac{2}{n\pi \sinh \frac{n\pi H}{L}} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx}$$

and A_0 remains arbitrary.

Part (c)

A_0 can be determined by considering the corresponding time-dependent problem with an initial condition.

$$\begin{aligned}
&\frac{\partial u}{\partial t} = k\nabla^2 u, \quad 0 < x < L,\ 0 < y < H,\ t > 0 \\
&\frac{\partial u}{\partial x}(0, y) = 0 \\
&\frac{\partial u}{\partial x}(L, y) = 0 \\
&\frac{\partial u}{\partial y}(x, 0) = 0 \\
&\frac{\partial u}{\partial y}(x, H) = f(x) \\
&u(x, y, 0) = g(x, y)
\end{aligned}$$

Integrate both sides of the PDE over the area A of the rectangle.

$$\iint_A \frac{\partial u}{\partial t}\, dA = \iint_A k\nabla^2 u\, dA$$

Bring the time derivative in front of the integral on the left. It becomes a total derivative, as the double integral wipes out the x and y variables. Apply Green's theorem to the double integral on the right.

$$\begin{aligned}
\frac{d}{dt} \iint_A u(x, y, t)\, dA &= k \iint_A \nabla \cdot \nabla u\, dA \\
&= k \oint_{\text{bdy } A} \nabla u \cdot \mathbf{n}\, dr
\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}\iint\limits_A u(x,y,t)\,dA &= k\left[\int_0^L \frac{\partial u}{\partial y}(x,0)\,dx + \int_0^H \frac{\partial u}{\partial x}(L,y)\,dy + \int_L^0 \frac{\partial u}{\partial y}(x,H)\,dx + \int_H^0 \frac{\partial u}{\partial x}(0,y)\,dy\right] \\ &= k\left[\int_0^L (0)\,dx + \int_0^H (0)\,dy + \int_L^0 f(x)\,dx + \int_H^0 (0)\,dy\right] \\ &= -k\int_0^L f(x)\,dx\end{aligned}$$

For there to be an equilibrium temperature distribution, the right side must be zero (the solvability condition).

$$\frac{d}{dt}\iint\limits_A u(x,y,t)\,dA = 0$$

Integrate both sides with respect to t.

$$\iint\limits_A u(x,y,t)\,dA = \text{constant}$$

The double integral on the left is the same regardless of what time is chosen.

$$\iint\limits_A u(x,y,0)\,dA = \iint\limits_A u(x,y,\infty)\,dA$$

$$\begin{aligned}\int_0^H\int_0^L g(x,y)\,dx\,dy &= \int_0^H\int_0^L \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}\cosh\frac{n\pi y}{L}\right)dx\,dy \\ &= \int_0^H\int_0^L A_0\,dx\,dy + \int_0^H\int_0^L \sum_{n=1}^{\infty} A_n\cos\frac{n\pi x}{L}\cosh\frac{n\pi y}{L}\,dx\,dy \\ &= \int_0^H\int_0^L A_0\,dx\,dy + \sum_{n=1}^{\infty} A_n \underbrace{\left(\int_0^L \cos\frac{n\pi x}{L}\,dx\right)}_{=0}\left(\int_0^H \cosh\frac{n\pi y}{L}\,dy\right)\end{aligned}$$

$$\int_0^H\int_0^L g(x,y)\,dx\,dy = A_0(HL)$$

Therefore,

$$\boxed{A_0 = \frac{1}{HL}\int_0^H\int_0^L g(x,y)\,dx\,dy.}$$

A_0 is the average of the initial temperature distribution over the area.